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STABILITY OF FLOW OF A LIQUID CRYSTAL LAYER ON AN INCLINED PLANE

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The framework of the linear mechanics of liquid crystal media [1] is used to study propagation of waves in a layer of a nematic liquid crystal (NLC) on an inclined plane, in a magnetic field, for three different cases of orientation of the anisotropy axis, namely orthogonal to the inclined plane, parallel to the inclined plane and orthogonal to the plane of flow. Such orientations of the anisotropy axis are realized in practice in the course of special machining of solid surfaces [2]. Exact solutions of the equations of motion are obtained describing the steady flow of the layer, and the behavior of small plane perturbations is studied. It is shown that two types of plane waves can propagate in a layer of the nematic mesophase, namely, the surface and the orientational waves. In the case of long surface waves the formulas for the critical Reynolds number are obtained. For the orientational waves a sufficient criterion of stability of the flow in the layer is obtained for two cases. The influence of the magnetic field and of the rheological parameters of NLC on the character of propagation of the first and second type waves is investigated.

From amongst the papers dealing with wave propagation in NLC, we draw the readers' attention to [3] which deals with the longitudinal, shear and torsional waves in a liquid crystal domain and obtains the corresponding dispersion relationships.

1. Equations of motion of NLC. Let us write the equations of motion of the incompressible nematic mesophase in the dimensionless form, for the case when the angle between the axes of nematic orientation is small [1]

$$\frac{d^2L_2}{dt^2} = -E_1M_1 - \frac{E_1B_5}{R^2}\frac{\partial}{\partial x_1}\left(\frac{\partial L_1}{\partial x_2} - \frac{\partial L_2}{\partial x_1}\right) + \frac{E_1B_1}{R^2}\Delta L_2 + \frac{E_1B_4}{R^2}\frac{\partial^2 L_2}{\partial x_3^2} + \tag{1.1}$$
$$\frac{E_1}{2R}(\delta_3+\delta_4-\delta_2)\left(\frac{\partial v_2}{\partial x_3} - \frac{dL_2}{dt}\right) - \frac{E_1}{2R}(\delta_2+\delta_4-\delta_3)\left(\frac{\partial v_3}{\partial x_2} + \frac{dL_2}{dt}\right)$$

$$\frac{d^2L_1}{dt^2} = E_1M_2 + \frac{E_1B_5}{R^2}\frac{\partial}{\partial x_2}\left(\frac{\partial L_1}{\partial x_2} - \frac{\partial L_2}{\partial x_1}\right) + \frac{E_1B_1}{R^2}\Delta L_1 + \frac{E_1B_4}{R^2}\frac{\partial^2 L_1}{\partial x_3^2} +$$
$$\frac{E_1}{2R}(\delta_3+\delta_4-\delta_2)\left(\frac{\partial v_1}{\partial x_3} - \frac{dL_1}{dt}\right) - \frac{E_1}{2R}(\delta_2+\delta_4-\delta_3)\left(\frac{\partial v_3}{\partial x_1} + \frac{dL_1}{dt}\right)$$

$$\frac{d\omega}{dt} = E_2M_3 + \frac{E_2\delta_9}{R}\Delta\omega + \frac{E_2\delta_{10}}{R}\frac{\partial^2\omega}{\partial x_3^2} + \frac{E_2\delta_5}{R}\cdot\left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} - 2\omega\right)$$

$$\frac{dv_1}{dt} = F_1 - \frac{\partial p}{\partial x_1} + \frac{1}{R}\Delta v_1 + \frac{\delta_5}{R}\frac{\partial\omega}{\partial x_2} + \frac{1}{R}\frac{\partial}{\partial x_3}\Big[\delta_3\frac{\partial v_1}{\partial x_3} + (\delta_6-\delta_7)\times$$
$$\frac{\partial v_3}{\partial x_1} - \frac{1}{2}(\delta_3+\delta_4-\delta_2)\frac{dL_1}{dt}\Big]$$

$$\frac{dv_2}{dt} = F_2 - \frac{\partial p}{\partial x_2} + \frac{1}{R}\Delta v_2 - \frac{\delta_5}{R}\frac{\partial \omega}{\partial x_1} +$$
$$\frac{1}{R}\frac{\partial}{\partial x_3}\left[\delta_3 \frac{\partial v_2}{\partial x_3} + (\delta_6 - \delta_7)\frac{\partial v_3}{\partial x_2} - \frac{1}{2}(\delta_3 + \delta_4 - \delta_2)\frac{dL_2}{dt}\right]$$
$$\frac{dv_3}{dt} = F_3 - \frac{\partial p}{\partial x_3} + \frac{\delta_2}{R}\Delta v_3 + \frac{\delta_8 - \delta_6}{R}\frac{\partial^2 v_3}{\partial x_3^2} + \frac{1}{2R}(\delta_2 + \delta_4 - \delta_3)\times$$
$$\left[\frac{\partial}{\partial x_1}\frac{dL_1}{dt} + \frac{\partial}{\partial x_2}\frac{dL_2}{dt}\right]$$
$$\delta_j = \eta_j / \eta_1, \quad E_1 = h^2 J_\perp^{-1}, \quad E_2 = h^2 J_\parallel^{-1}$$
$$R = \rho v_a h \eta_1^{-1}, \quad B_1 = \rho d_{1212}\eta_1^{-2}, \quad B_2 = \rho d_{1221}\eta_1^{-2}$$
$$B_3 = \rho d_{1122}\eta_1^{-2}, \quad B_4 = \rho d_{1313}\eta_1^{-2}, \quad B_5 = B_2 + B_3$$
$$M_i = m_i v_a^{-2}, \quad F_i = f_i h v_a^{-2}, \quad \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$
$$i = 1, 2, 3; \qquad j = 2, 3, \ldots, 10$$

Here x_1, x_2, x_3 and t are the dimensionless Cartesian coordinates and time; v_i are the components of the velocity vector and p is pressure relative to v_a and ρv_a^2, respectively; L_i are the components of the unit anisotropy vector, ω is the mean rate of rotation of molecules about their central axes parallel to L and relative to $v_a h^{-1}$; η_1, $\eta_2, \ldots, \eta_{10}$ are the viscosity coefficients; $J_\perp$ and $J_\parallel$ are constants characterizing the local moments of inertia of the medium; d_{1212}, d_{1221}, d_{1122} and d_{1313} are the moduli of elasticity; ρ is density, v_a is the characteristic velocity; h is the characteristic dimension; f_i and m_i are the components of the volume force vector and the volume momentum, respectively.

The equations of the linear mechanics of NLC given above are written in the coordinate system, the axis x_3 of which is directed along the L-axis in the undeformed state [1]. Therefore in all the cases which follow the homogeneous magnetic field of strength H which preserves the orientation of the L-axis, will be assumed as directed along the x_3-axis.

2. Anisotropy axis orthogonal to the inclined plane. Let us consider a layer of nematic mesophase of thickness h, flowing down the inclined plane under the gravity. We assume that near the solid surface the longitudinal axes of the NLC molecules are oriented along the normal to this surface. Let us introduce the right Cartesian x_1, x_2, x_3 coordinate system with the origin at the layer surface, the x_2- and x_3-axes directed along a line of steepest descent and inside the fluid, respectively. Then the equations of motion (1.1) admit the following exact solution describing a steady flow in the layer:

$$v_2^\circ = {}^3/_2(1 - x_3^2), \quad p^\circ = 3\delta_3 R^{-1} x_3 \operatorname{ctg}\gamma + p_a(\rho v_a^2)^{-1} \qquad (2.1)$$
$$L_2^\circ = {}^3/_2 \varkappa^{-1} R(\delta_3 + \delta_4 - \delta_2)\,[(m \operatorname{ch} m)^{-1}(m - \operatorname{sh} m)\operatorname{ch} mx_3 + m^{-1}\operatorname{sh} mx_3 - x_3]$$
$$\omega^\circ = 0, \quad \varkappa = \Delta\chi\eta_1^{-2}H^2\rho h^2, \quad \Delta\chi = \chi_\parallel - \chi_\perp$$
$$m = hH(\Delta\chi d_{1313}^{-1})^{1/2}, \qquad v_a = {}^1/_3 \rho g \eta_3^{-1} h^2 \sin\gamma$$

Here γ is the angle of inclination of the plane to the horizon, p_a is the atmospheric pressure, g is acceleration due to gravity and $\Delta\chi$ is the anisotropy of the magnetic susceptibility. The solution (2.1) is obtained under the boundary conditions of adhesion at the solid surface and the absence of the torque and shearing stresses at the free surface.

Let us investigate the stability of the steady flow with respect to small periodic perturbations.

We note that the Squire's theorem [4] holds for the orientation of the anisotropy axis defined above. To prove this it is sufficient to write the equations of motion linearized in the usual manner [5] and the boundary conditions for the NLC layer in a coordinate system obtained from the initial system by rotating it about the x_3-axis in such a manner that the x_2'-axis of the new system is orthogonal to the expansion front of the three-dimensional wave. At the same time we find that the equations controlling the stability correspond to a plane flow of the layer with the velocity profile $v_2^\circ \cos\beta$, where β is the angle between the x_2- and x_2'-axes. For this reason we only need to consider the plane perturbations while investigating the stability.

Let us denote by ω', L_1', L_2', u', v' and w' the corresponding perturbations in the natural angular velocity of rotation of the molecules and the projections of the unit anisotropy vector and the velocity vector on the x_1, x_2, x_3 coordinate axes. It can easily be shown that in the case of plane perturbations the linearized equations of motion and the boundary conditions can formally be separated into two independent groups defining the behavior of the perturbations v', w', L_2' and u', L_1', ω', respectively. From the physical point of view this means that two unconnected kinds of plane waves may propagate in the layer of the nematic mesophase. Let us set

$$v' = \frac{\partial\psi}{\partial x_3}, \quad w' = -\frac{\partial\psi}{\partial x_2}, \quad \psi = \varphi(x_3)\, e^{i\alpha(x_2 - c_1 t)} \tag{2.2}$$
$$L_2' = L_2^*(x_3)\, e^{i\alpha(x_2 - c_1 t)}, \quad L_1' = L_1^*(x_3)\, e^{i\alpha(x_2 - c_2 t)}$$
$$u' = u^*(x_3)\, e^{i\alpha(x_2 - c_2 t)}, \quad \omega' = \omega^*(x_3)\, e^{i\alpha(x_2 - c_2 t)}$$

Here φ, L_2^*, L_1^*, u^* and ω^* are complex amplitudes of the perturbations, ψ is the stream function and α is a dimensionless wave number, while c_1 and c_2 are the complex velocities of propagation of the first and second type waves. Then we have the following two boundary value problems for the complex wave amplitudes

$$\frac{d^4\varphi}{dx_3^4} - \delta_3^{-1}(1 - 2\delta_6 + \delta_7 + \delta_8)\alpha^2 \frac{d^2\varphi}{dx_3^2} + \delta_2\delta_3^{-1}\alpha^4\varphi = \tag{2.3}$$
$$i\alpha\delta_3^{-1}R\left[(v_2^\circ - c_1)\left(\frac{d^2\varphi}{dx_3^2} - \alpha^2\varphi\right) - \varphi\frac{d^2 v_2^\circ}{dx_3^2}\right] +$$
$$\frac{1}{2}i\alpha\delta_3^{-1}\left\{(\delta_3 + \delta_4 - \delta_2)\left[\frac{d^2L_2^*}{dx_3^2}(v_2^\circ - c_1) + 2\frac{dL_2^*}{dx_3}\frac{dv_2^\circ}{dx_3} + \right.\right.$$
$$\left. L_2^*\frac{d^2v_2^\circ}{dx_3^2} - \left(\varphi\frac{d^3L_2^\circ}{dx_3^3} + 2\frac{d\varphi}{dx_3}\frac{d^2L_2^\circ}{dx_3^2} + \frac{d^2\varphi}{dx_3^2}\frac{dL_2^\circ}{dx_3}\right)\right] +$$
$$\left.\alpha^2(\delta_2 + \delta_4 - \delta_3)\left[\varphi\frac{dL_2^\circ}{dx_3} - L_2^*(v_2^\circ - c_1)\right]\right\}$$
$$L_2^*[\varkappa + \alpha^2B_1 + i\alpha R\delta_4(v_2^\circ - c_1)] - B_4\frac{d^2L_2^*}{dx_3^2} - i\alpha R\delta_4\varphi\frac{dL_2^\circ}{dx_3} -$$
$$\frac{1}{2}R(\delta_3 + \delta_4 - \delta_2)\frac{d^2\varphi}{dx_3^2} + \frac{1}{2}\alpha^2R(\delta_2 + \delta_4 - \delta_3)\varphi = 0$$
$$\varphi(1) = \frac{d\varphi(1)}{dx_3} = L_2^*(1) = 0, \quad \frac{d^2L_2^\circ(0)}{dx_3^2}\frac{\varphi(0)}{c_1 - v_2^\circ(0)} + \frac{dL_2^*(0)}{dx_3} = 0$$
$$2\delta_3\left(\frac{d^2\varphi(0)}{dx_3^2} + \frac{d^2v_2^\circ(0)}{dx_3^2}\frac{\varphi(0)}{c_1 - v_2^\circ(0)}\right) + i\alpha^2(\delta_2 + \delta_3 - \delta_4)\varphi(0) +$$
$$i\alpha(\delta_3 + \delta_4 - \delta_2)(c_1 - v_2^\circ(0))L_2^*(0) = 0$$

$$\alpha(3\delta_3 \operatorname{ctg}\gamma + \alpha^2 S')\frac{\varphi(0)}{c_1 - v_2^\circ(0)} + \alpha R\left[\varphi(0)\frac{dv_2^\circ(0)}{dx_3} - (v_2^\circ(0) - c_1)\frac{d\varphi(0)}{dx_3}\right] +$$
$$\frac{1}{2}i\alpha^2(2 + \delta_2 + \delta_3 - \delta_4 - 4\delta_6 + 2\delta_7 + 2\delta_8)\frac{d\varphi(0)}{dx_3} - i\delta_3\frac{d^3\varphi(0)}{dx_3^3} -$$
$$\frac{1}{2}\alpha(\delta_3 + \delta_4 - \delta_2)\left[\frac{dL_2^*(0)}{dx_3}(v_2^\circ(0) - c_1) + L_2^*(0)\frac{dv_2^\circ(0)}{dx_3} -\right.$$
$$\left.\left(\varphi(0)\frac{d^2L_2^\circ(0)}{dx_3^2} + \frac{d\varphi(0)}{dx_3}\frac{dL_2^\circ(0)}{dx_3}\right)\right] = 0, \quad S' = RS$$

$$i\alpha u^*(v_2^\circ - c_2) = R^{-1}\left(\delta_3\frac{d^2u^*}{dx_3^2} - \alpha^2 u^*\right) + i\alpha\delta_5 R^{-1}\omega^* - \qquad (2.4)$$
$$\frac{1}{2}i\alpha R^{-1}(\delta_3 + \delta_4 - \delta_2)\left[\frac{dL_1^*}{dx_3}(v_2^\circ - c_2) + L_1^*\frac{dv_2^\circ}{dx_3}\right]$$
$$L_1^*[\varkappa + \alpha^2(B_1 + B_5) + i\alpha R\delta_4(v_2^\circ - c_2)] - B_4\frac{d^2L_1^*}{dx_3^2} -$$
$$\frac{1}{2}R(\delta_3 + \delta_4 - \delta_2)\frac{du^*}{dx_3} = 0$$
$$\delta_{10}\frac{d^2\omega^*}{dx_3^2} - \alpha^2\delta_9\omega^* - \delta_5(i\alpha u^* + 2\omega^*) = 0$$
$$u^*(1) = L_1^*(1) = \omega^*(1) = \frac{dL_1^*(0)}{dx_3} = \frac{d\omega^*(0)}{dx_3} = 0$$
$$2\delta_3\frac{du^*(0)}{dx_3} + i\alpha(\delta_3 + \delta_4 - \delta_2)(c_2 - v_2^\circ(0))L_1^*(0) = 0$$

Here S is the coefficient of surface tension relative to $\rho v_a^2 h$. In the course of deriving (2.3) and (2.4) it was assumed that $J_\perp = J_\parallel = 0$, which is physically justified for the NLC media [1] by virtue of the smallness of the local moment of inertia.

Equations (2.3) correspond to the v', w', L_2' perturbation wave which by its nature, is analogous to a surface perturbation wave in an ordinary viscous fluid. On the contrary, the u', L_1', ω' perturbation wave has no relation to any distortion of the free surface and is of predominantly orientational character. In a wave of this type the oscillatory translations of the molecules in the x_1 direction vary periodically with respect to x_2 and time. The molecules rotate about their long axes of inertia as well as about the x_2 direction. The possibility of propagation of such waves depends essentially on the presence of the rotational degrees of freedom in the liquid crystal and on the anisotropic structure of the NLC media.

Let us study the behavior of the long-wave surface perturbations, using the method of consecutive approximations [6, 7] to solve the boundary value problem (2.3). Restricting ourselves to the first two approximations, we write φ, L_2^* and c_1 in the form

$$\varphi = \varphi_0 + \alpha\varphi_1, \quad L_2^* = l_0 + \alpha l_1, \quad c_1 = c_1^\circ + \alpha c_1' \qquad (2.5)$$

Substituting (2.5) into (2.3) we obtain the following expressions for the zero approximation:

$$\varphi_0 = (x_3 - 1)^2, \quad l_0 = R(\varkappa \operatorname{ch} m)^{-1}(\delta_3 + \delta_4 - \delta_2) \times \qquad (2.6)$$
$$\{[(m - \operatorname{sh} m)\operatorname{th} m - 1]\operatorname{ch} mx_3 - (m - \operatorname{sh} m)\operatorname{sh} mx_3 + \operatorname{ch} m\}$$
$$c_1^\circ = 3$$

The next approximation gives the following expression for c_1':

$$c_1' = i\{\delta_3^{-1}R[{}^9/_8(\varkappa \operatorname{ch} m)^{-1}(\delta_3 + \delta_4 - \delta_2)^2\Phi(m) + {}^6/_5] - \operatorname{ctg}\gamma\} \qquad (2.7)$$
$$\Phi(m) = {}^1/_3 m^3 D_4 + m^2 D_3 - m \operatorname{sh} m(D_1 + D_3 + D_5) + mD_2 -$$
$$(\operatorname{ch} m + {}^1/_2 m^2 - 1)(D_1 - D_5) - m \operatorname{ch} m(D_2 + D_4) + D_4 \operatorname{sh} m$$

$$\begin{aligned}
D_1 &= m^{-2}\,[(m - \mathrm{sh}\, m)\mathrm{th}\, m + \mathrm{ch}\, m - 1] \\
D_2 &= -4m^{-3}\,[(m - \mathrm{sh}\, m)\,(\mathrm{th}\, m + {}^1/_2 m) + \mathrm{ch}\, m - 1] \\
D_3 &= -2m^{-2}\,\mathrm{ch}\, m, \qquad D_4 = 4m^{-3}\,\mathrm{ch}\, m \\
D_5 &= (6 + m^2)m^{-4}\,[(m - \mathrm{sh}\, m)\mathrm{th}\, m + \mathrm{ch}\, m - 1] + 4m^{-3}\,(m - \mathrm{sh}\, m)
\end{aligned}$$

This at once yields the formula for the critical Reynolds number

$$R_1^* = \delta_3\,[1 + {}^{15}/_{16}(B_4 m^2\,\mathrm{ch}\, m)^{-1}\,(\delta_3 + \delta_4 - \delta_2)^2\Phi\,(m)]^{-1}R_0^* \qquad (2.8)$$

where $R_0^* = {}^5/_6\,\mathrm{ctg}\,\gamma$ is the critical Reynolds number for a layer of an isotropic Newtonian fluid [6].

Let us find the order of quantities entering (2.8). As we know from [1], the case when the angle between the axes of the nematic order is small, requires a sufficiently large magnetic field. Since for the NLC media we have $\Delta\chi \sim 10^{-6}$ cm^3/g and $d_{1313} \sim 10^{-6}$ dynes, then $m \sim H$ and must also be large. It can easily be shown that when m increases, the expression $(m^2\,\mathrm{ch}\, m)^{-1}\,\Phi\,(m) \to -0$ as m^{-2}. Consequently, assuming that the viscosity $\eta_1 \sim 10^{-2}$ poises, we obtain $B_4 \sim 10^{-2}$ which implies that a magnetic field of several hundred oersteds is sufficient for $|\,(B_4 m^2\,\mathrm{ch}\, m)^{-1}\,\Phi\,(m)\,| < 1$. Since by virtue of the anisotropic character of viscosity of the NLC $\delta_3 > 1$ [1, 2], the coefficient accompanying R_0^* in (2.8) is always greater than unity. This means that the flow of the NLC layer under the specified orientation of the anisotropy axis is always more stable with respect to the surface perturbations than the flow of a layer of Newtonian fluid. An increase in the strength of magnetic field reduces the value of R_1^*. At the same time $\lim\limits_{H\to\infty} R_1^* = \delta_3 R_0^*$, which agrees with the result obtained in [8], namely, that when $H \to \infty$, the flow of nematic medium becomes identical to that of an ordinary Newtonian fluid of viscosity η_3.

The destabilizing influence of the magnetic field on the behavior of the surface perturbations can be explained as follows. The hydrodynamic flow exerting a significant orienting influence on the structure of the NLC causes a nonuniform orientation of the molecules across the layer thickness. In the case when the angle between the axes of nematic order is small, which was considered above, the deviation of the orientation of the liquid crystal molecules in a steady flow from the orientation prevailing near the solid surface, is determined by the quantity L_2°. On the contrary, the magnetic field acting in the direction parallel to the anisotropy axis near the wall, tends to produce a uniform orientation of the molecules right across the flow and thus exerts a competing influence. This leads to reduction in the hydrodynamic stability of the flow with respect to the surface perturbations.

From (2.8) it follows that increasing the modulus of elasticity d_{1313} reduces the value of R_1^*. Consequently the elasticity of NLC affects the stability of the flow just as the magnetic field does. On the contrary, increasing the value of the viscosity coefficient η_3 stabilizes the flow.

Let us turn our attention to the boundary value problem (2.4) for the amplitudes of the orientational type perturbations. Formal application of the algorithm of the asymptotic expansions [6, 7] in the wave number α to the system (2.4) does not produce the desired result, as the zero, first and all further approximations to the perturbation amplitudes are identically zero. However, using the approach analogous to the Synge's method in the theory of hydrodynamic stability of a viscous fluid [4], we can obtain a sufficient condition for the stability of the NLC flow relative to the orientational waves. Let us multiply the second equation of (2.4) by $\bar{L}_1^*$, which is the complex conjugate of L_1^*, and integrate the result in x_3 from zero to one. Separating the real part of the resulting expression we obtain

$$\delta_4 I_1 c_2^i = \alpha^{-1}\left\{\frac{1}{4}(\delta_3+\delta_4-\delta_2)\int_0^1\left(\frac{du^*}{dx_3}\bar{L}_1^*+\frac{d\bar{u}^*}{dx_3}L_1^*\right)dx_3 - R^{-1}\left[\varkappa I_1+\alpha^2(B_1+B_5)I_1+B_4 I_2\right]\right\} \tag{2.9}$$

$$I_1=\int_0^1 |L_1^*|^2 dx_3, \qquad I_2=\int_0^1\left|\frac{dL_1^*}{dx_3}\right|^2 dx_3$$

Here c_2^i is the imaginary part of the complex wave velocity c_2. Using the estimate

$$\left|\int_0^1\left(\frac{du^*}{dx_3}\bar{L}_1^*+\frac{d\bar{u}^*}{dx_3}L_1^*\right)dx_3\right|\leqslant 2I_3, \qquad I_3=\int_0^1\left|\frac{du^*}{dx_3}\right||L_1^*|\,dx_3$$

we obtain from (2.9) the following condition of attenuation of the orientational waves

$$R<\min\left\{\frac{2\left[\varkappa+\alpha^2(B_1+B_5)\right]I_1+2B_4 I_2}{(\delta_3+\delta_4-\delta_2)I_3}\right\} \tag{2.10}$$

Thus, when the Reynolds number is sufficiently small, the orientational type perturbations attenuate. In addition, as we see from (2.9), the magnetic field and the elasticity of the liquid crystal are stabilizing factors. Consequently the magnetic field as well as the elasticity of NLC exert a reverse influence on the development of the corresponding surface and orientational type perturbations.

3. Axis of anisotropy parallel to the inclined plane. Consider the case when the axis of anisotropy of NLC is situated in the plane of flow near the solid surface, and is parallel to this surface. Let the x_3-axis be directed along the line of the steepest descent, and the x_1-axis along the normal to the solid surface inside the fluid. Using this coordinate system we find, that the steady flow of the layer is described by the following solution of the equations of motion (1.1):

$$v_3^\circ = {}^3/_2(1-x_1^2), \qquad p^\circ = 3\delta_2 R^{-1}x_1 \operatorname{ctg}\gamma + p_a(\rho v_a^2)^{-1} \tag{3.1}$$

$$L_1^\circ = {}^3/_2\varkappa^{-1}R(\delta_2+\delta_4-\delta_3)\left[m_1(\operatorname{ch} m_1)^{-1}(\operatorname{sh} m_1 - m_1)\operatorname{ch} m_1 x_1 - m_1^{-1}\operatorname{sh} m_1 x_1 + x_1\right]$$

$$\omega^\circ = 0, \quad m_1 = hH(\Delta\chi d_{1212}^{-1})^{1/2}, \quad v_a = {}^1/_3\rho g\eta_2^{-1}h^2\sin\gamma$$

Let us study the behavior of the small perturbations independent of the coordinate x_2. As in the previous case in which the axis of anisotropy is orthogonal to the inclined plane, the linearized equations of motion and the boundary conditions for the perturbations can be separated into two mutually independent groups corresponding to the surface waves and the orientational waves. Retaining the previous notation for the perturbations, we set

$$u'=\frac{\partial\psi}{\partial x_3}, \quad w'=-\frac{\partial\psi}{\partial x_1}, \quad \psi=\varphi(x_1)e^{i\alpha(x_3-c_1 t)}$$
$$L_1'=L_1^*(x_1)e^{i\alpha(x_3-c_1 t)}, \qquad L_2'=L_2^*(x_1)e^{i\alpha(x_3-c_2 t)} \tag{3.2}$$
$$v'=v^*(x_1)e^{i\alpha(x_3-c_2 t)}, \qquad \omega'=\omega^*(x_1)e^{i\alpha(x_3-c_2 t)}$$

and again construct two boundary value problems, the problem corresponding to the surface u', w', L_1' perturbation wave and the problem corresponding to the orientational v', L_2', ω' perturbation wave. If we take the equations of the boundary value problem for the amplitudes of the surface perturbations and replace the parameters d_{1212}, η_2 by d_{1313}, η_3 and vice versa, the resulting equations become the boundary value problem (2.3). Therefore, using the results of Sect. 2, we obtain the following expression for the critical Reynolds number of the flow in question:

$$R_2^* = \delta_2 [1 + {}^{15}/_{16} (B_1 m_1^2 \operatorname{ch} m_1)^{-1} (\delta_2 + \delta_4 - \delta_3)^2 \Phi (m_1)]^{-1} R_0^* \quad (3.3)$$

The effect of the magnetic field and the modulus of elasticity on the above relation was discussed previously. We only note that $\lim_{H\to\infty} R_2^* = \delta_2 R_0^*$ which also agrees with the result of [8].

The boundary value problem for the amplitudes of the orientational perturbations cannot be reduced to Eqs. (2.4), and must be considered separately. The problem has the form

$$(B_5 + B_1) \frac{d^2 L_2^*}{dx_1^2} - [\varkappa + \alpha^2 B_4 + i\alpha R \delta_4 (v_3^\circ - c_2)] L_2^* + \frac{1}{2} i\alpha R (\delta_3 + \delta_4 - \delta_2) v^* = 0 \quad (3.4)$$

$$\delta_9 \frac{d^2\omega^*}{dx_1^2} - (\alpha^2 \delta_{10} + 2\delta_5) \omega^* + \delta_5 \frac{dv^*}{dx_1} = 0 \quad (3.5)$$

$$\frac{d^2 v^*}{dx_1^2} - [\alpha^2 \delta_3 + i\alpha R (v_3^\circ - c_2)] v^* - \delta_5 \frac{d\omega^*}{dx_1} + \frac{1}{2} \alpha^2 (\delta_3 + \delta_4 - \delta_2)(v_3^\circ - c_2) L_2^* = 0 \quad (3.6)$$

$$L_2^*(1) = v^*(1) = \omega^*(1) = \frac{d\omega^*(0)}{dx_1} = \frac{dL_2^*(0)}{dx_1} = 0 \quad (3.7)$$
$$dv^*(0)/dx_1 - \delta_5 \omega^*(0) = 0$$

Applying to (3.4) the case discussed in Sect. 2, we obtain the following sufficient criterion of stability of the flow with respect to the orientational waves:

$$\alpha R < \min \left\{ \frac{2 [(B_5 + B_1) I_4 + (\varkappa + \alpha^2 B_4) I_5]}{(\delta_3 + \delta_4 - \delta_2) I_6} \right\} \quad (3.8)$$

$$I_4 = \int_0^1 \left| \frac{dL_2^*}{dx_1} \right|^2 dx_1, \quad I_5 = \int_0^1 | L_2^* |^2 dx_1, \quad I_6 = \int_0^1 | v^* | | L_2^* | dx_1$$

From (3.8) it follows that an interval of variation in the values of α and R for which the orientational type perturbations decay, always exists. The dimensions of this interval increase with the increasing values of the moduli of elasticity of NLC and the magnetic field strength.

Let us consider the behavior of the long orientational waves in the case when the coefficients of rotational viscosity δ_5 is negligibly small [1]. Multiplying (3.6) by $\bar{v}^*$ and integrating in x_1 from zero to one, with the boundary conditions taken into account, we obtain (neglecting the terms containing α^2):

$$\alpha R I_8 c_2^i = - I_7 - i\alpha R \int_0^1 (v_3^\circ - c_2^r) | v^* |^2 dx_1 \quad (3.9)$$

$$I_7 = \int_0^1 \left| \frac{dv^*}{dx_1} \right|^2 dx_1, \qquad I_8 = \int_0^1 | v^* |^2 dx_1$$

Here c_2^r is the real part of the wave velocity c_2. The imaginary part of (3.9) gives

$$\alpha R \int_0^1 (v_3^\circ - c_2^r) | v^* |^2 dx_1 = 0$$

from which it follows that the difference $v_3^\circ - c_2^r$ changes its sign in the interval (0, 1). Consequently the velocity of propagation of the orientational type wave satisfies, in this case, the inequality $0 < c_2^r < 3/2$. Separating the real part of (3.9) we obtain

$$c_2^i = -I_7 (\alpha R I_8)^{-1} < 0$$

Thus, when the rotational viscosity is absent, the long orientational waves always decay. We also note that when $\delta_5 = 0$, Eq. (3.5) and the boundary conditions (3.7) together imply that $\omega^* = 0$, i.e. in this case the orientational waves do not induce the rotational oscillations of the molecules.

4. Axis of anisotropy orthogonal to the plane of flow. The steady flow of the layer is described by the following relations:

$$\begin{aligned} v_1^\circ &= K \{1 - x_2^2 + \delta_5 n^{-2} [(\mathrm{ch}\, n)^{-1} (n - \mathrm{sh}\, n)\, \mathrm{sh}\, n x_2 + \\ &\quad \mathrm{ch}\, n x_2] - \delta_5 (n^2\, \mathrm{ch}\, n)^{-1} (n\, \mathrm{sh}\, n + 1)\} \\ \omega^\circ &= K [x_2 - (n\, \mathrm{ch}\, n)^{-1} (n - \mathrm{sh}\, n)\, \mathrm{ch}\, n x_2 - n^{-1}\, \mathrm{sh}\, n x_2] \\ p^\circ &= (2 - \delta_5) R^{-1} K x_2\, \mathrm{ctg}\, \gamma + p_a (\rho v_a^2)^{-1} \\ K &= [2/3 + \delta_5 n^{-2} (1 - 2 (\mathrm{ch}\, n)^{-1} + (n\, \mathrm{ch}\, n)^{-1}\, \mathrm{sh}\, n\, (1 - n^2))]^{-1} \\ n &= [\delta_5 \delta_9^{-1} (2 - \delta_5)]^{1/2}, \quad v_a = [\eta_1 K (2 - \delta_5)]^{-1} \rho g h^2 \sin \gamma \end{aligned} \tag{4.1}$$

The solution (4.1) is written in the coordinate system in which the x_1-axis is directed along the line of steepest descent and the x_2-axis is directed into the layer. Assuming that the small perturbations are independent of the coordinate x_3, we linearize the equations of motion and the boundary conditions for the layer. Setting

$$\begin{aligned} u' &= \frac{\partial \psi}{\partial x_2}, \quad v' = -\frac{\partial \psi}{\partial x_1}, \quad \psi = \varphi(x_2)\, e^{i\alpha (x_1 - c_1 t)} \\ \omega' &= f(x_2)\, e^{i\alpha (x_1 - c_1 t)}, \quad w' = w^*(x_2)\, e^{i\alpha (x_1 - c_2 t)} \\ L_1' &= L_1^*(x_2)\, e^{i\alpha (x_1 - c_2 t)}, \quad L_2' = L_2^*(x_2)\, e^{i\alpha (x_1 - c_2 t)} \end{aligned} \tag{4.2}$$

we obtain the following equations for the complex amplitudes:

$$\begin{aligned} &\frac{d^4\varphi}{dx_2^4} - 2\alpha^2 \frac{d^2\varphi}{dx_2^2} + \alpha^4 \varphi = i\alpha R \left[(v_1^\circ - c_1) \left(\frac{d^2\varphi}{dx_2^2} - \alpha^2 \varphi \right) - \right. \\ &\quad \left. \varphi \frac{d^2 v_1^\circ}{dx_2^2} \right] + \delta_5 \left(\alpha^2 f - \frac{d^2 f}{dx_2^2} \right) \\ &\delta_9 \left(\frac{d^2 f}{dx_2^2} - \alpha^2 f \right) + \delta_5 \left(\alpha^2 \varphi - \frac{d^2\varphi}{dx_2^2} - 2f \right) = 0 \\ &\varphi(1) = \frac{d\varphi(1)}{dx_2} = f(1) = 0 \end{aligned} \tag{4.3}$$

$$\left(\frac{d^2v_1^\circ(0)}{dx_2^2}+\delta_5\frac{d\omega^\circ(0)}{dx_2}\right)\frac{\varphi(0)}{c_1-v_1^\circ(0)}+$$
$$\frac{d^2\varphi(0)}{dx_2^2}+\alpha^2(1-\delta_5)\varphi(0)+\delta_5 f(0)=0$$

$$[K(2-\delta_5)\operatorname{ctg}\gamma+\alpha^2S']\frac{\varphi(0)}{c_1-v_1^\circ(0)}+R\varphi(0)\frac{dv_1^\circ(0)}{dx_2}-$$
$$R(v_1^\circ(0)-c_1)\frac{d\varphi(0)}{dx_2}-i\alpha^{-1}\frac{d^3\varphi(0)}{dx_2^3}-i\delta_5\alpha^{-1}\frac{df(0)}{dx_2}+$$
$$i\alpha(3-\delta_5)\frac{d\varphi(0)}{dx_2}=0,\quad \frac{d^2\omega^\circ(0)}{dx_2^2}\frac{\varphi(0)}{c_1-v_1^\circ(0)}+\frac{df(0)}{dx_2}=0$$

$$\frac{d^2w^*}{dx_2^2}-\alpha^2w^*-i\alpha R\delta_2^{-1}(v_1^\circ-c_2)w^*+\frac{1}{2}\delta_2^{-1}(\delta_2+\delta_4-\delta_3)\times \tag{4.4}$$
$$\left[\left(i\alpha\frac{dL_2^*}{dx_2}-\alpha^2L_1^*\right)(v_1^\circ-c_2)+i\alpha L_2^*\frac{dv_1^\circ}{dx_2}\right]=0$$

$$L_2^*[\varkappa+\alpha^2(B_5+B_1)+i\alpha R\delta_4(v_1^\circ-c_2)]+i\alpha B_5\frac{dL_1^*}{dx_2}-$$
$$B_1\frac{d^2L_2^*}{dx_2^2}+\frac{1}{2}R(\delta_2+\delta_4-\delta_3)\frac{dw^*}{dx_2}=0$$

$$L_1^*[\varkappa+\alpha^2B_1+i\alpha R\delta_4(v_1^\circ-c_2)]+i\alpha B_5\frac{dL_2^*}{dx_2}-$$
$$(B_1+B_5)\frac{d^2L_1^*}{dx_2^2}+\frac{1}{2}i\alpha R(\delta_2+\delta_4-\delta_3)w^*=0$$

$$L_1^*(1)=L_2^*(1)=w^*(1)=0,\quad B_1\frac{dL_2^*(0)}{dx_2}-i\alpha B_2L_1^*(0)=0$$
$$2\delta_2\frac{dw^*(0)}{dx_2}+i\alpha(\delta_2+\delta_4-\delta_3)(v_1^\circ(0)-c_2)L_2^*(0)=0$$
$$(B_1+B_5)\frac{dL_1^*(0)}{dx_2}-i\alpha B_3L_2^*(0)=0$$

The system (4.3) describes the behavior of the surface perturbations and (4.4) the behavior of the orientational perturbations. We note that when $\delta_5=0$, the first equation of (4.3) becomes the Orr-Sommerfeld equation and the boundary conditions for φ assume the form of those in the problem of the stability of flow of a layer of a viscous Newtonian fluid [6].

Let us construct the solution of the boundary value problem (4.3) for the case of the long waves. Writing φ, f and c_1 in the form of series in powers of α, we obtain the following zero order approximation for the rate of propagation of the surface perturbations:

$$c_1^\circ=v_1^\circ(0)+K(\operatorname{ch}n)^{-1}[\operatorname{ch}n+\delta_5n^{-2}(\operatorname{ch}n-1)-$$
$$\delta_5n^{-1}\operatorname{sh}n]-K\delta_5(n\operatorname{ch}n)^{-2}(n-\operatorname{sh}n)^2$$

The next approximation gives the expression for the critical Reynolds number

$$R_3^*=G(\delta_5,\ \delta_9)\operatorname{ctg}\gamma$$

where $G(\delta_5,\delta_9)$ is a certain function of the parameters δ_5 and δ_9. Thus the critical Reynolds number of flow for surface waves depends only on the coefficients of rotational and moment viscosities δ_5 and δ_9 and is independent of the magnetic field strength. This is connected with the fact that under the present orientation of the axis of anisotropy (4.3) implies that the surface perturbation wave is not accompanied by a distortion in the oriented structure of the layer.

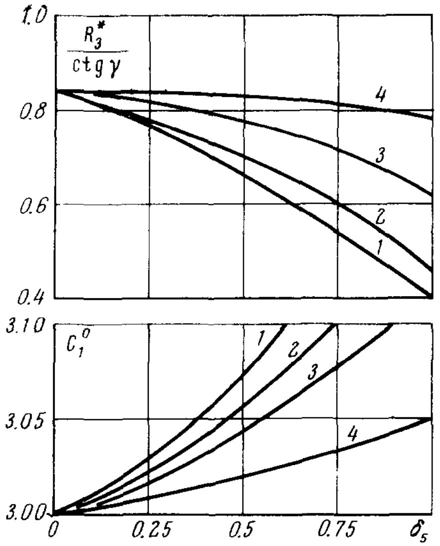

Fig. 1

Numerical computations were performed on the computer BESM-4 in order to clarify the influence of the parameters δ_5 and δ_9 on R_3^* and c_1°, and the results are depicted in Fig. 1. The curves 1–4 correspond to the values of $\delta_9 = 0.01,\ 0.1,\ 1$, and 10. The curves show that the rotational viscosity has a destabilizing effect on the flow, while the moment viscosity enhances stability. When $\delta_5 = 0$, we have $R_3^* = {}^5/_6\, \mathrm{ctg}\,\gamma$ which corresponds to the value of the critical Reynolds number for a layer of the Newtonian fluid. The velocity of propagation of the surface wave c_1° increases with increasing δ_5 and decreases with increasing δ_9. Moreover, when $\delta_5 \neq 0$, we always have $c_1^\circ > 3$.

In conclusion we note that the method of long wave approximations [6, 7] cannot be applied to the system (4.4), nor to the systems (2.4) and (3.4)–(3.7). We also cannot obtain for this particular orientation of the axis of anisotropy, the sufficient criterion of stability for the orientational waves of the type (2.10) and (3.8). For this reason the boundary value problem (4.4) requires additional investigation.

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